Stability and Mean Growth of Stochastic Solow Growth Models with Jump and Regime-switching

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Outline

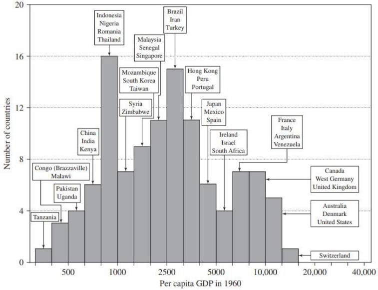
1. **Background**

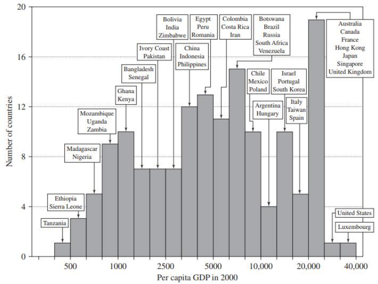
– *The Importance of Growth*

– *Neoclassical Growth Model of Solow*

1-I: The Importance of Growth

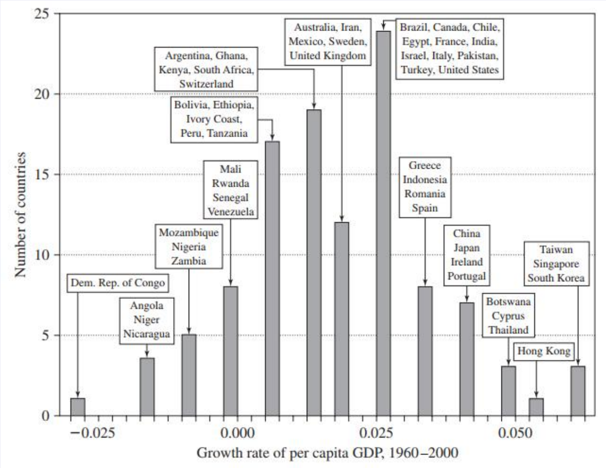
Per Capita GDP, 1960 / 2000 (from *Penn World Tables*)





1-I: The Importance of Growth

Growth rate of per capita GDP, 1960 - 2000, (from *Penn World Tables*)



1-I: The Importance of Growth

▷ Why are some countries rich and others poor? What are the sources of recessions and booms?

▷ Why countries experience such sharp divergences in long-term growth rates?

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▷ Why are some countries rich and others poor? What are the sources of recessions and booms?

▷ Why countries experience such sharp divergences in long-term growth rates?

▷ Even small differences in these growth rates, when cumulated over 40 years or more, have **much greater consequences for standards of living** than the kinds of short-term business fluctuations.

1-II: Neoclassical Growth Model of Solow

▷ Historical Line:

Input-output method (Leontief, 1973 Nobel Prize)

Neoclassical (新古典) growth model (Solow, 1987 Nobel Prize)

→ **Solow model with uncertainty** (Merton, 1997 Nobel Prize)

→ Endogenous (内生) (human capital) growth model (Lucas, 1995 Nobel Prize)

→ Endogenous (内生) (technology) growth model (Romer, 2018 Nobel Prize)

1-II: Neoclassical Growth Model of Solow

The Solow model focuses on four variables: output (总产出, Y_t), capital (资本, K_t), labor (劳动力, L_t), and technology (技术, A_t). The economy has some amounts of capital, labor, and technology, and these are combined to produce output:

$$Y_t = F(K_t, A_t L_t) = A_t L_t \cdot F(K_t/(A_t L_t), 1). \quad \text{(规模报酬不变)}$$

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Assume that the dynamic of main variables are described by differential equations

$$dK_t/dt = [Y_t - C_t] - \delta K_t = sY_t - \delta K_t,$$

$$dL_t/dt = hL_t,$$

$$dA_t/dt = gA_t.$$

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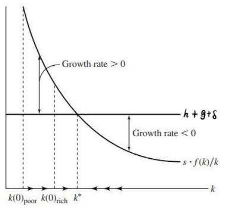
Denote $k_t = K_t/(A_t L_t)$, $y_t = Y_t/(A_t L_t)$ and $f(k_t) = F(k_t, 1)$ then $y_t = f(k_t)$, then we obtain the key equation of the Solow model

$$\frac{dk_t}{dt} = sf(k_t) - (h + g + \delta)k_t.$$

1-II: Neoclassical Growth Model of Solow

Growth rate of Solow model: $k_t'/k_t = sf(k_t)/k_t - (h+g+\delta)$



▷ The balanced growth path k^*. How the variables behave when k_t equals k^*?

$$\frac{dK_t/dt}{K_t} = \frac{dY_t/dt}{Y_t} = h + g.$$

Capital per worker, K_t/L_t, and output per worker, Y_t/L_t, are growing at rate g.

Solow: Capital accumulation is not the main factor of economic growth, **long-term economic growth must depend on population growth and technological progress (increase in education)**.

2. Solow Growth Model with Uncertainty

- *The Study of Uncertainty*
- *Stochastic Solow Model with Jump*
- *Stochastic Solow Model on Random Environments*

2-I: The Study of Uncertainty

▷ Importance of uncertainty:

Merton (*1975, Rev Econ Stud*): introduce uncertainty into Solow model, since then, more and more scholars realize **the effect of uncertainty on the stability and long time behavior** of economic systems;

▷ AK-type growth model: Steger (*2005, Econ Lett*), Buera and Oberfield (*2015, Econometrica*), Boucekkine et al. (*2018, Econ Rev*);

▷ Solow growth model: Ramsey and Ramsey (*1995, Am Econ Rev*), Wu and Hu (*2009, J Appl Probab*), Bloom et al. (*2015, Am Econ Rev*);

▷ Endogenous growth model: Hek (*1999, Int Econ Rev*), Canton (*2002, Econ Theor*), Bucci et al. (*2011, J Econ*), Tsuboi (*2020, Econ Model*).

2-I: The Study of Uncertainty

▷ The conflicting effects of uncertainty:

▷ **Positive role:**

Canton (*2002, Econ Theor*) proved that economic growth is higher since people devote more time in scientific research in an uncertain economic environment;

Wu and Hu (*2009, J Appl Probab*) illustrated the positive effect of variances of population growth and capital accumulation through the exponential instability of the stochastic differential equations.

2-I: The Study of Uncertainty

▷ The conflicting effects of uncertainty:

▷ **Negative role:**

Hek (*1999, Int Econ Rev*) examined uncertainty in the creation of knowledge and constructed a corresponding endogenous growth model to explain the negative effects of uncertainty;

Ramsey and Ramsey (*1995, Am Econ Rev*) empirical analysed of economic development data for 92 countries yields that countries with higher uncertainty have lower economic growth rates;

Tsuboi (*2020, Econ Model*) solved the stochastic R&D model and concluded that the uncertainty tends to severely restrict the growth of income.

2-I: The Study of Uncertainty

▷ Why:

▷ i. Uncertainty arises for a variety of reasons, and the impact of different types of uncertainty varies.

▷ ii. There is an interaction between various uncertainties.

▷ iii. The economic environment changes with policies, wars, disasters, etc., and uncertainty plays a different role in different environments.

2-I: The Study of Uncertainty

The main work:

▷ Three types of uncertainty into the Solow growth model:

▷ (1). From the process of technology:

▷ Continuous part i: fluctuation produced by the accumulation of technologies (driven by Brownian motion B_t);

▷ Jump part ii: spurt of improvement driven by new inventions and ideas (driven by Poisson random measure $N(t, dz)$);

▷ (2). From the macro-environment variabilities, such as policies, wars, disasters

▷ The environmental variation (described by Markov chain Λ_t).

2-I: The Study of Uncertainty

Difficulties and main contributions:

▷ Develop a more realistic stochastic model of economic growth.

▷ (1). Give the **criterion** of the stochastic stability of the system; propose the concept of balanced mean growth path.

▷ (2). **Quantitatively analyze** how these three uncertainties affect the main variables of the economic system.

▷ (3). Give **computable methods** for calculating the mean growth rates, balanced mean growth path, moment estimations, convergence rate, asymptotic boundedness, etc.

2-II: Stochastic Solow Model with Jump

The technology A_t is an adaptive stochastic processes given by

$$dA_t = gA_t dt + \sigma_1 A_t dB_t + \sigma_2 A_{t-} \int_{\mathbb{R}_0} zN(dt, dz), \quad A_0 = a_0 > 0,$$

where σ_1 and σ_2 are the variances of continuous part and jump part, when $\sigma_2 = 0$, the definition of A_t covers the model given in *Tsuboi (2020)*.

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The dynamics of capital and labor can be characterized by equations

$$\begin{cases} dK_t = (sY_t - \delta K_t)dt, & K_0 = k_0 > 0, \\ dL_t = hL_t dt, & L_0 = l_0 > 0, \end{cases}$$

where $s \in [0, 1]$ is the saving rate, $\delta \in [0, 1]$ is the depreciation rate of capital.

2-II: Stochastic Solow Model with Jump

Capital, labor and technology, as factors of production, are transformed into the total output in a certain way. The input-output relationship is characterized by production function, which is **Cobb-Douglas production function** (柯布-道格拉斯生产函数), i.e.,

$$Y_t = F(K_t, A_t L_t) = K_t^{\alpha}(A_t L_t)^{1-\alpha}, \quad 0 < \alpha < 1,$$

where α, $1 - \alpha$ are the marginal outputs of capital and effective labor.

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Let $X_t = K_t/(L_t A_t)$ be the effective capital-labor ratio, then

$$dX_t = (sX_t^{\alpha} + mX_t)\,dt - \sigma_1 X_t dB_t - \int_{\mathbb{R}_0} X_{t-}\left(\frac{\sigma_2 z}{1+\sigma_2 z}\right) N(dt, dz),$$

where $m = \sigma_1^2 - g - h - \delta$. Easy to verify that $x^* = 0$ is a solution corresponding to the initial condition $X_0 = 0$, which is called the **trivial solution**. If $x^* = 0$ is stable, X_t will converge to zero and **the economy will be in stagnant state**.

2-II: Stochastic Solow Model with Jump

The growth rate and steady state are the most important concepts.

Definition: mean growth rate & balanced mean growth path

For each stochastic process X_t, define a process by stochastic differential equation

$$dG(X_t) = X_{t-}^{-1} dX_t.$$

Then defien

$$\text{mean growth: } G_X(t) := \mathbb{E}\left[G(X_t)\right], \quad \text{mean growth rate: } G'_X(t) := \frac{d}{dt}\mathbb{E}\left[G(X_t)\right].$$

The balanced mean growth path π^* is a distribution on $\mathbb{R}_+$, such that $G'_X(t) = 0$ when $X_t \sim \pi^*$ for some $t \geqslant 0$.

Remark: balanced mean growth path v.s. stationary distribution.

2-III: Stochastic Solow Model on Random Environments

The component Λ_t is a Markov chain takes values in $\mathbb{S} = \{1, 2, \ldots, n\}$, with $n < \infty$. The transfer rate matrix is denoted by $Q = (q_{ij})_{n \times n}$, which is assumed irreducible and conservative.

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The component X_t is the effective capital-labor ratio in Λ_t satisfying

$$dX_t = \left(s(\Lambda_t) X_t^{\alpha(\Lambda_t)} + m(\Lambda_t) X_t \right) dt - \sigma_1(\Lambda_t) X_t dB_t - \int_{\mathbb{R}_0} X_{t-} \left(\frac{\sigma_2(\Lambda_{t-}) z}{1 + \sigma_2(\Lambda_{t-}) z} \right) N(dt, dz),$$

with initial condition $X_0 = x_0 \in \mathbb{R}_0$ and $\Lambda_0 = i_0 \in \mathbb{S}$.

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with initial condition $X_0 = x_0 \in \mathbb{R}_0$ and $\Lambda_0 = i_0 \in \mathbb{S}$.

The main aim of this part is to explore the influence on stability due to B_t, $N(t, A)$ and Λ_t, respectively.

3. Results I: Stability and Mean Growth in Fixed Environment

- *Stability and Moment Estimation of Capital-Labor Ratio*
- *Estimation of Mean Growth*
- *Balanced Mean Growth Path*

3-I: Stability and Moment Estimation of Capital-Labor Ratio

Theorem (i) - stability of $x = 0$

The trivial solution $x = 0$ is almost surely exponentially **unstable** if

$$\beta := m - \frac{1}{2}\sigma_1^2 - \lambda C_1(\sigma_2) > 0,$$

where $C_1(\sigma_2) := \int_{\mathbb{R}_0} \log(1 + \sigma_2 z)\varphi(dz)$.

Remark: We also have the criterions of stochastic stability/instability.

3-I: Stability and Moment Estimation of Capital-Labor Ratio

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Theorem (ii) - pth moment of X_t

Under the condition of Theorem (i), let $p \geqslant 1$, the pth moment satisfies

$$\mathbb{E}\left[X_t^p\right] \leqslant e^{tM_p} x_0^p + \left(e^{tM_p} - 1\right),$$

where $C(\kappa, \sigma_2) := \int_{\mathbb{R}_0} \left[(1 + \sigma_2 z)^\kappa - 1\right] \varphi(dz)$ and

$$M_p = s(p + \alpha - 1)\left(\frac{p + \alpha - 1}{1 - \alpha}\right)^{\frac{\alpha - 1}{p}} + \left(mp + \frac{1}{2}p(p-1)\sigma_1^2 + \lambda C(-p, \sigma_2)\right) > 0.$$

3-II: Estimation of Mean Growth

Theorem (iii) - on mean growth of X_t

Under the condition of Theorem (i). The mean growth rate of X_t:

$$m + \lambda C(-1, \sigma_2) \leqslant G'_X(t) \leqslant s x_0^{\alpha-1} e^{tM_\alpha} + m + \lambda C(-1, \sigma_2),$$

where $M_\alpha := (\alpha - 1)\left[m - \frac{\sigma_1^2}{2}(2-\alpha) + \frac{\lambda}{\alpha-1} C(1-\alpha, \sigma_2)\right]$. Particularly, there exists $0 < \alpha^*(\gamma_X) < 1$ such that $M_\alpha < 0$ when $\alpha^*(\gamma_X) < \alpha < 1$.

Remark: The marginal output should not be too large, otherwise it will **affect the long time behavior of the system**. Denote by $\bar{X}_t := K_t/L_t$, we have

Theorem (iv) - on mean growth of economic variables

$$G'_K(t) \leqslant x_0^{\alpha-1}\left(s e^{tM_\alpha}\right) - \delta, \qquad G'_{\bar{X}}(t) \leqslant x_0^{\alpha-1}\left(s e^{tM_\alpha}\right) - \delta - h$$

$$G'_Y(t) \leqslant x_0^{\alpha-1}\left(\alpha s e^{tM_\alpha}\right) + (1-\alpha)\left[\frac{-\alpha\delta}{1-\alpha} + h + g - \frac{1}{2}\alpha\sigma_1^2 + \frac{\lambda}{1-\alpha} C(1-\alpha, \sigma_2)\right].$$

III: Balanced Mean Growth Path

Theorem (iv) - on balanced mean growth path

Under the condition of Theorem (i) and $m + \lambda C(-1, \sigma_2) < 0$. The distribution π^* of the balanced mean growth path satisfies

$$\int_{\mathbb{R}_0} x^{\alpha-1} \pi^*(dx) = -s^{-1} \left(m + \lambda C(-1, \sigma_2) \right).$$

Moreover, when $X_{t^*} \sim \pi^*$ for some $t^* \geqslant 0$, the mean growth rate of capital K_t, total output Y_t and capital-labor ratio $\bar{X}_t$ satisfy

$$G_K'(t^*) = h + g - \sigma_1^2 + \lambda \int_{\mathbb{R}_0} \left(\frac{\sigma_2 z}{1 + \sigma_2 z} \right) \varphi(dz),$$

$$G_Y'(t^*) = h + g - \frac{1}{2}\alpha(3 - \alpha)\sigma_1^2 + \lambda \int_{\mathbb{R}_0} \left[\left(\frac{\alpha \sigma_2 z}{1 + \sigma_2 z} \right) + (1 + \sigma_2 z)^{1-\alpha} - 1 \right] \varphi(dz),$$

$$G_{\bar{X}}'(t^*) = g - \sigma_1^2 + \lambda \int_{\mathbb{R}_0} \left(\frac{\sigma_2 z}{1 + \sigma_2 z} \right) \varphi(dz).$$

4. Results II: Stochastic Solow Models with Regime-Switching

– *Recurrent and Transient*

– *Convergence Rate of Stationary Distribution*

– *Asymptotic Boundedness of pth Moment*

4-I: Recurrent and Transient

The regime-switching process (X_t, Λ_t) is given by

$$dX_t = \left(s(\Lambda_t)X_t^{\alpha(\Lambda_t)} + m(\Lambda_t)X_t\right)dt - \sigma_1(\Lambda_t)X_t dB_t - \int_{\mathbb{R}_0} X_{t-}\left(\frac{\sigma_2(\Lambda_{t-})z}{1+\sigma_2(\Lambda_{t-})z}\right)N(dt, dz),$$

with initial condition $X_0 = x_0 \in \mathbb{R}_0$ and $\Lambda_0 = i_0 \in \mathbb{S}$.

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Thmeorem (v) - on recurrence

Let $\mu = (\mu_i)_{i\in\mathbb{S}}$ be the stationary distribution of Λ_t. Denote by β_i as

$$\beta_i = m(i) - \frac{1}{2}\sigma_1^2(i) - \lambda C_1(\sigma_2(i)).$$

Then,

(i). the process (X_t, Λ_t) is positive recurrent if $\sum_{i\in\mathbb{S}} \mu_i\beta_i < 0$.

(ii). the process (X_t, Λ_t) is transient if $\sum_{i\in\mathbb{S}} \mu_i\beta_i > 0$.

4-II: Convergence Rate of Stationary Distribution

Under the condition of of positive recurrence, there exists a unique stationary distribution $\pi(dx, i)$ on $\mathbb{R}_0 \times \mathbb{S}$ such that the semigroup P_t of (X_t, Λ_t) converges weakly to $\pi(dx, i)$ as $t \to \infty$ for every initial data $(x_0, i_0) \in \mathbb{R}_0 \times \mathbb{S}$.

Theorem (vi) - on convergence rate

Under the conditions of positive recurrence in Theorem (v), denote $\widehat{\beta} := \sum_{i \in \mathbb{S}} \beta_i \mu_i < 0$. Then, there exist constants $C > 0$ and $0 < p < 1$ such that

$$d_{TV}(\delta_{(x_0,i_0)} P_t, \pi) \leqslant C e^{(p\widehat{\beta})t} (1 + x_0^p), \quad \forall t \geqslant 0, \ (x_0, i_0) \in \mathbb{R}_0 \times \mathbb{S}.$$

4-III: Asymptotic Boundedness of pth Moment

▷ In our economic model with regime-switching, the mean growth rate is completely determined by the current environment $i \in \mathbb{S}$ of the economic system, and the calculation methods of $G'_K(t)$, $G'_Y(t)$ and $G'_{\bar{X}}(t)$ are similar.

▷ In practical applications, it is likely that the economic system does not satisfy the condition of positive recurrence, however, using the M-matrix method, we will be able to obtain the asymptotic behavior of process (X_t, Λ_t), such as the **asymptotic boundedness of pth moment**.

4-III: Asymptotic Boundedness of pth Moment

Definition 4.3 - M-matrix

A is M-matrix if it can be expressed as $A = \lambda I - B$ with non-negative matrix B and $\lambda > \rho(B)$, where I is identity matrix and $\rho(B)$ is the spectral radius of B.

Theorem 4.4 - on pth moment estimations

Let $p \geqslant 1$ and denote by $\Gamma := -\text{diag}\big(M(p,1), \cdots, M(p,n)\big) - Q$, where

$$M(p,i) := p\left\{m(i) + \frac{1}{2}(p-1)\sigma_1^2(i) + \frac{\lambda}{p}C(-p, \sigma_2(i))\right\},$$

If Γ is a nonsingular M-matrix, there exists positive numbers λ, κ and η^* (calculable),

$$\limsup_{t\to\infty} \mathbb{E}\left[X_t^p\right] \leqslant \frac{\kappa}{\eta^*\lambda}.$$

5. **Example: System in Negative and Positive Environment**

– *Negative Environment*

– *Positive Environment*

– *Model with Regime-Switching*

Economic System in Negative and Positive Environment

Consider an economic system that switches between two environments, i.e., $\mathbb{S} = \{0, 1\}$. To eliminate the interference of secondary variables, we assume that

$$h(0) = h(1) = 0, \quad \delta(0) = \delta(1) = 0.$$

Moreover, we also assume that

$$\alpha(0) = \alpha(1) = \alpha \in (0, 1), \quad g(0) = g(1) = g \geqslant 0, \quad s(0) = s(1) = s \in [0, 1],$$

The switching is described by a Markov chain Λ_t, which is generated by matrix Q with stationary distribution μ,

$$Q = \begin{pmatrix} -r_0 & r_0 \\ r_1 & -r_1 \end{pmatrix}, \quad \mu = (\mu_0, \mu_1) = \left(\frac{r_1}{r_0 + r_1}, \frac{r_0}{r_0 + r_1} \right). \tag{1}$$

In our example, we assume that the economic system consists of two distinct environments, one is **negative environment** (denoted by $i = 0$) and the other is **positive environment** (denoted by $i = 1$).

5-I: Negative Environment - $i = 0$

▷ Technology in this environment **lacks the motive force and incentive mechanism of innovation**, but it is **subject to greater stochastic disturbance** in the process of accumulation.

We assume that $\sigma_1(0) = \sigma_1 > 0$ and $\sigma_2(0) = 0$. Moreover, assume that the stochastic disturbance is sufficiently large, that is

$$\frac{1}{2}\alpha\sigma_1^2 > g. \qquad (\star)$$

5-I: Negative Environment - $i = 0$

According to theorems on stability and mean growth, pth-moment and balanced path:

▷ $X_t^{(0)} \to \infty$ as $t \to \infty$ almost surely (since $A_t^{(0)} \to 0$ as $t \to \infty$).

▷ Since $m(0) + \lambda C(-1, \sigma_2(0)) = \sigma_1^2 - g > 0$, the mean growth rate $G'_{X^{(0)}}(t)$ is always positive and there is no balanced growth path for $X_t^{(0)}$;

According to theorems on mean growth of economic variables,

▷ Since $M_\alpha(0) = (1-\alpha)\left[g - \frac{\alpha}{2}\sigma_1^2\right] < 0$, the long-run mean growth rates of capital $K_t^{(0)}$, total output $Y_t^{(0)}$ and capital-labor ratio $\bar{X}_t^{(0)}$ satisfy

$$\lim_{t\to\infty} G'_{K^{(0)}}(t) = \lim_{t\to\infty} G'_{\bar{X}^{(0)}}(t) = 0, \quad \lim_{t\to\infty} G'_{Y^{(0)}}(t) = (1-\alpha)\left[g - \frac{\alpha}{2}\sigma_1^2\right] < 0.$$

5-I: Negative Environment - $i = 0$

Therefore, in this environment $i = 0$, the economic system will eventually **collapse due to excessive stochastic disturbance driven by Brownian motion**, that is, capital and capital-labor ratio will stagnate, and the total output even produces negative growth.

In other words, **stochastic disturbances** B_t in technology have a **negative impact** on the economic system.

5-II: Positive Environment - $i = 1$

▷ Technological innovation and research are **greatly encouraged and supported**.

We assume that $\sigma_2(1) = \sigma_2 > 0$ and the intensity $\lambda\varphi(dz)dt$ satisfies that

$$\lambda \int_{\mathbb{R}_0} \left(\frac{\sigma_2 z}{1 + \sigma_2 z} \right) \varphi(dz) > \sigma_1^2 - g, \qquad (\star\star)$$

where $\sigma_1(1) = \sigma_1(0) = \sigma_1 > 0$ with $(\star)$ holds.

5-II: Positive Environment - $i = 1$

▷ The trivial solution $a = 0$ of $A_t^{(1)}$ is almost exponentially unstable and $A_t^{(1)}$ has a positive mean growth rate, since

$$g - \frac{1}{2}\sigma_1^2 > \frac{1}{2}\sigma_1^2 - \lambda \int_{\mathbb{R}_0} \left(\frac{\sigma_2 z}{1 + \sigma_2 z} \right) \varphi(dz) > -\lambda \int_{\mathbb{R}_0} \log\left(1 + \sigma_2 z\right) \varphi(dz).$$

According to theorems on mean growth of economic variables,

▷ The parameter $M_\alpha(1)$ is

$$M_\alpha(1) = (1 - \alpha) \left[g - \frac{\alpha}{2}\sigma_1^2 + \frac{\lambda}{1 - \alpha} \int_{\mathbb{R}_0} \left((1 + \sigma_2 z)^{1-\alpha} - 1 \right) \varphi(dz) \right] > 0.$$

The upper and lower bound of mean growth rates satisfies,

$$G'_{K(1)}(t) \text{ and } G'_{\bar{X}(1)}(t) \in \left(0,\ s x_0^{\alpha-1} e^{t M_\alpha(1)}\right],$$
$$G'_{Y(1)}(t) \in \left(M_\alpha(1),\ \alpha s x_0^{\alpha-1} e^{t M_\alpha(1)} + M_\alpha(1)\right].$$

5-II: Positive Environment - $i = 1$

According to theorem on balanced mean growth path,

▷ There exists a balanced mean growth path π^* and when $X_{t^*}^{(1)} \sim \pi^*$ for some $t^* \geqslant 0$, and the mean growth rates of $K_t^{(1)}$, $Y_t^{(1)}$ and $\bar{X}_t^{(1)}$ on the balanced mean growth path satisfies

$$G'_{K^{(1)}}(t^*) = G'_{\bar{X}^{(1)}}(t^*) = g - \sigma_1^2 + \lambda \int_{\mathbb{R}_0} \left(\frac{\sigma_2 z}{1 + \sigma_2 z} \right) \varphi(dz),$$

$$G'_{Y^{(1)}}(t^*) = g - \frac{1}{2}\alpha(3-\alpha)\sigma_1^2 + \lambda \int_{\mathbb{R}_0} \left[\frac{\alpha \sigma_2 z}{1 + \sigma_2 z} + (1 + \sigma_2 z)^{\alpha - 1} \right] \varphi(dz).$$

It should also be noted that the increase in variance σ_2 is of limited help to the economy, but **it is more important to increase parameter λ and $\varphi(dz)$**, i.e. the impact of inventions and creations.

5-II: Positive Environment - $i = 1$

Therefore, in this environment $i = 1$, the **Poisson random measure** has a **positive role** and it **contributes to the stability and growth rates of the main variables of the economic system**. In other words, the jump in technology driven by new invention has a very prominent and positive impact on the economic system.

Moreover, formula $(\star\star)$ and the above discussion show that when the jump part is strong enough, its positive effect will cancel out the negative effect caused by the Brownian motion.

5-III: Model with Regime-Switching

In this part, we will discuss the process (X_t, Λ_t) with regime-switching in the negative and positive environments. The parameters of this system satisfy that $\sigma_1(0) = \sigma_1(1) = \sigma_1$ with $(\star)$, $\sigma_2(0) = 0$ and $\sigma_2(1) = \sigma_2$ with $(\star\star)$.

According to Theorem 4.1 (on recurrence of switching process), we have

$$\text{Negative environment:} \quad \beta_0 = \frac{1}{2}\sigma_1^2 - g > 0,$$

$$\text{Positive environment:} \quad \beta_1 = \frac{1}{2}\sigma_1^2 - g - \lambda \int_{\mathbb{R}_0} \log(1 + \sigma_2 z)\varphi(dz) < 0.$$

5-III: Model with Regime-Switching

To ensure that the process $(\widehat{X}_t, \Lambda_t)$, as well as (X_t, Λ_t), is positive recurrent, we need

$$\widehat{\beta} := \mu_0\beta_0 + \mu_1\beta_1$$
$$= \left(\frac{1}{r_0 + r_1}\right)\left[\left(\frac{1}{2}\sigma_1^2 - g\right)r_1 + \left(\frac{1}{2}\sigma_1^2 - g - \lambda\int_{\mathbb{R}_0}\log(1+\sigma_2 z)\varphi(dz)\right)r_0\right] < 0,$$

which is equivalent to

$$\frac{1}{2}\sigma_1^2 - g < \left(\frac{r_0}{r_0 + r_1}\right)\cdot\lambda\int_{\mathbb{R}_0}\log(1+\sigma_2 z)\varphi(dz). \qquad (\star\star\star)$$

Define $\varepsilon^* := -\beta_1/\beta_0 > 0$, then above formula holds when

$$r_0 > r_1/\varepsilon^*.$$

5-III: Model with Regime-Switching

According to the definition of Q matrix, the sojourn time of Λ_t on state $i = 0$ satisfies

$$\mathbb{P}[\Lambda_s = 0, 0 \leqslant s \leqslant t | \Lambda_0 = 0] = e^{-r_0 t}.$$

Therefore, once r_0 is sufficiently large (equivalently, **the sojourn time of state $i = 0$ is short**), the process $(\widehat{X}_t, \Lambda_t)$ is still positive recurrent.

The discussion here shows that despite the existence of some negative environment in the process of economic development, **we can control and regulate the switching mechanism between the environments so that the overall economic system remains stable**. In this sense, the switching part may serve as a stabilizing factor.

5-III: Model with Regime-Switching

Finally, we analyze the asymptotic boundedness of (X_t, Λ_t). Consider $p = 1$, then

$$M_0 := M(1,0) = \sigma_1^2 - g,$$
$$M_1 := M(1,1) = \sigma_1^2 - g - \lambda \int_{\mathbb{R}_0} \left(\frac{\sigma_2 z}{1 + \sigma_2 z} \right) \varphi(dz).$$

The matrix Γ in theorem of moment estimation equals

$$\Gamma = \begin{pmatrix} -M_0 + r_0 & -r_0 \\ -r_1 & -M_1 + r_1 \end{pmatrix}.$$

Then Γ is a nonsingular M-matrix if and only if all of the principal minors of Γ are positive, that is

$$r_0 - M_0 > 0 \quad \text{and} \quad (r_1 - M_1)(r_0 - M_0) - r_0 r_1 > 0.$$

Since $(\star)$ and $(\star\star)$, we obtain $M_0 > 0$ and $M_1 < 0$. Above inequalities hold if

$$r_0 > M_0 - (r_1 M_0)/M_1.$$

5-III: Model with Regime-Switching

Define a constant φ with

$$\varphi = \frac{r_0(r_0 + r_1 - M_0)}{(r_0 - M_0)(r_1 + r_0 - M_1)} < 1.$$

According to theorem on pth-moment estimations, we can calculate $\kappa/(\eta^*\lambda)$ and then

▷ the asymptotic bound of X_t is

$$\begin{aligned}\limsup_{t\to\infty} \mathbb{E}\left[X_t\right] &\leqslant \left(\frac{r_0}{\varphi}\right)\left(\frac{s}{1-\varphi}\right)^{\frac{1}{1-\alpha}} (r_0 - M_0)^{-\frac{2-\alpha}{1-\alpha}} \\ &= \left(\frac{r_0 + r_1 - M_1}{r_0 + r_1 - M_0}\right)\left(\frac{s(r_0 + r_1 - M_1)}{M_0 M_1 - r_0 M_1 - M_0 r_1}\right)^{\frac{1}{1-\alpha}}.\end{aligned}$$

Thank you for your attention!